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An anisotropic magnetohydrodynamic cosmological model in general relativity

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An anisotropic magnetohydrodynamic cosmological model has been derived which is expanding and shearing but is nonrotating.

1. Derivation of the Line-Element

We consider the metric of space-time for the cosmological model in the form (Marder 1958)

$$ds^2 = A^2(dx^2 - dt^2) + B^2dy^2 + C^2dz^2 \qquad \dots \quad (1.1)$$

where A, B, C are functions of t alone. This is a transform of the metric of Bianchi Type I space-time in comoving coordinates which has been studied by a number of authors e.g. (Heckmann & Schucking 1962) and (Thorne 1967). In this paper we have considered distribution of matter to consist of an electrically neutral perfect fluid with an infinite electrical conductivity and a magnetic field. The energy momentum tensor of the composite field is assumed to be the sum of the corresponding energy momentum tensors. Thus

$$T_i^j = (\epsilon + p)v_iv^j + p\delta_i^j + E_i^j \qquad \dots \quad (1.2)$$

where

$$E_i^j = \mu[h_lh^l(v_iv^j + \tfrac{1}{2}\delta_i^j) - h_ih^j] \qquad \dots \quad (1.3)$$

μ being the magnetic permeability, ϵ the density, p the pressure and $h_i = (1/\mu)$ v^jF_{ji} (Lichnerowicz 1967). F_{ij} is the electromagnetic field tensor and v^i is the flow vector satisfying

$$g_{ij}v^iv^j = -1 \qquad \dots \quad (1.4)$$

The coordinates are assumed to be comoving so that

$$v^1 = v^2 = v^3 = 0 \quad \text{and} \quad v^4 = \frac{1}{A}$$

We assume the incident magnetic field to be in the direction of the x-axis so that F_{23} is the only non vanishing component of the tensor F_{ij}. The first set of Maxwell's equation leads to F_{23} being a constant, say H. The field equations

$$R_i^j - \tfrac{1}{2}R\delta_i^j + \Lambda\delta_i^j = -8\pi T_i^j \qquad \dots \quad (1.5)$$

for the line-element (1.1) are as follows

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B}-\frac{C_{44}}{C}-\frac{B_4C_4}{BC}+\frac{A_4B_4}{AB}+\frac{A_4C_4}{AC}\right]-\Lambda=8\pi\left[p-\frac{H^2}{2\mu B^2C^2}\right] \quad \ldots \quad (1.6)$$

$$\frac{1}{A^2}\left[-\frac{C_{44}}{C}-\frac{A_{44}}{A}+\frac{A_4^2}{A^2}\right]-\Lambda=8\pi\left[p+\frac{H^2}{2\mu B^2C^2}\right] \quad \ldots \quad (1.7)$$

$$\frac{1}{A^2}\left[-\frac{B_{44}}{B}-\frac{A_{44}}{A}+\frac{A_4^2}{A^2}\right]-\Lambda=8\pi\left[p+\frac{H^2}{2\mu B^2C^2}\right] \quad \ldots \quad (1.8)$$

$$\frac{1}{A^2}\left[\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{A_4C_4}{AC}\right]+\Lambda=8\pi\left[\epsilon+\frac{H^2}{2\mu B^2C^2}\right] \quad \ldots \quad (1.9)$$

The suffix 4 after the symbols A, B, C indicates ordinary differentiation with respect to t. Eqs (1.6)–(1.9) are four equations in five unknowns A, B, C, ϵ and p. For the complete determination of these unknowns one more condition has to be imposed on them. Here we assume that the space-time is of degenerate Petrov type 1 the degeneracy being in y and z directions. This requires that $C_{12}{}^{12}=C_{13}{}^{13}$. This condition is identically satisfied if $B=C$. However we shall assume the metric potentials to be unequal owing to the assumed anisotropy. From eqs (1.6) and (1.7) we obtain

$$\left[\frac{A_4}{A}\right]_4+\frac{A_4}{A}\left[\frac{B_4}{B}+\frac{C_4}{C}\right]-\frac{B_{44}}{B}-\frac{B_4C_4}{BC}=-\frac{8\pi H^2A^2}{\mu B^2C^2}\,. \quad \ldots \quad (1.10)$$

Also, from eqs (1.7) and (1.8) we get

$$\frac{B_{44}}{B}=\frac{C_{44}}{C}\,. \quad \ldots \quad (1.11)$$

The condition $C_{12}{}^{12}=C_{13}{}^{13}$ leads to

$$\frac{B_{44}}{B}-\frac{C_{44}}{C}+\frac{2A_4}{A}\left[\frac{C_4}{C}-\frac{B_4}{B}\right]=0. \quad \ldots \quad (1.12)$$

From eqs (1.11) and (1.12) we obtain

$$\frac{A_4}{A}\left[\frac{C_4}{C}-\frac{B_4}{B}\right]=0. \quad \ldots \quad (1.13)$$

Since, $B\neq C$, equation (1.13) on integration gives

$$A=\text{constant}=N. \quad \ldots \quad (1.14)$$

From eqs (1.10) and (1.14), we get

$$\frac{B_{44}}{B}+\frac{B_4C_4}{BC}=\frac{8\pi H^2N^2}{\mu B^2C^2}. \qquad \text{... (1.15)}$$

Eq (1.11) on integration gives

$$B_4C-BC_4=K \qquad \text{... (1.16)}$$

K being a constant of integration. Putting $B/C=\alpha$ and $BC=\beta$ we have from eq (1.16)

$$\frac{\alpha_4}{\alpha}.\beta = K. \qquad \text{... (1.17)}$$

From eq. (1.15) we have

$$\frac{1}{\beta}\left[\left(\frac{\alpha_4}{\alpha}+\frac{\beta_4}{\beta}\right)\beta\right]_4=\frac{16\pi H^2N^2}{\mu\beta} \qquad \text{... (1.18)}$$

From equations (1.17) and (1.18) we get

$$\mu\beta\beta_{44}=16\pi H^2N^2 \qquad \text{... (1.19)}$$

which on integration gives

$$[\beta_4]^2=\frac{32\pi H^2N^2}{\mu}\log\beta+L \qquad \text{... (1.20)}$$

L being a constant of integration. From equations (1.17) and (1.20) we get

$$\alpha=b\exp\left[\frac{2K}{a^2}\{(a^2\log\beta+L)^{\frac{1}{2}}-L^{\frac{1}{2}}\}\right] \qquad \text{... (1.21)}$$

where $a^2=32\pi H^2N^2\,\dfrac{1}{\mu}$ and b is a constant of integration. Hence

$$B^2=b.\beta\exp\left[\frac{2K}{a^2}\{(a^2\log\beta+L)^{\frac{1}{2}}-L^{\frac{1}{2}}\}\right] \qquad \text{... (1.22)}$$

and

$$C^2=\frac{1}{b}.\beta\exp\left[-\frac{2K}{a^2}\{(a^2\log\beta+L)^{\frac{1}{2}}-L^{\frac{1}{2}}\}\right]. \qquad \text{... (1.23)}$$

Consequently the line-element (1.1) takes the form

$$ds^2 = N^2\left[dx^2 - \frac{d\beta^2}{(a^2\log\beta + L)}\right] + b\beta\exp\left[\frac{2K}{\alpha^2}\{(a^2\log\beta+L)^{\frac{1}{2}} - L^{\frac{1}{2}}\}\right]dy^2$$

$$+\frac{1}{b}\beta\exp\left[-\frac{2K}{a^2}\{(a^2\log\beta+L)^{\frac{1}{2}} - L^{\frac{1}{2}}\}\right]dz^2 \qquad \dots \quad (1.24)$$

By a suitable transformation of coordinates the metric (1.24) is reduced to the form

$$ds^2 = N^2\left[dx^2 - \frac{\exp(2T)}{(a^2T+L)}dT^2\right] + \exp\left[T + \frac{2K}{a^2}\{(a^2T+L)^{\frac{1}{2}} - L^{\frac{1}{2}}\}\right]d\gamma^2$$

$$+\exp\left[T - \frac{2K}{a^2}\{(a^2T+L)^{\frac{1}{2}} - L^{\frac{1}{2}}\}\right]dZ^2. \qquad \dots \quad (1.25)$$

2. Some Physical and Geometrical Features

The distribution in the model is given by

$$8\pi p = \frac{1}{4N^2}\exp(-2T)\left[L - K^2 + \frac{32\pi H^2N^2T}{\mu} - \frac{48\pi H^2N^2}{\mu}\right] - \Lambda \qquad \dots \quad (2.1)$$

$$8\pi\epsilon = \frac{1}{4N^2}\exp(-2T)\left[L - K^2 + \frac{32\pi H^2N^2T}{\mu} - \frac{16\pi H^2N^2}{\mu}\right] + \Lambda \qquad \dots \quad (2.2)$$

The charge current vector J given by

$$J^i = \left[\frac{1}{\mu}F^{ij}\right]_{,j}$$

turns out to be zero. The scalar of expansion θ is give by

$$\theta = \frac{1}{N\exp(T)}\left[L + \frac{32\pi H^2N^2T}{\mu}\right]^{\frac{1}{2}} \qquad \dots \quad (2.3)$$

The tensor of rotation ω_{ij} is zero and the components of the shear tensor σ_{ij} are given by

$$\sigma_{11} = -\frac{N}{3\exp(T)}\left[L + \frac{32\pi H^2N^2T}{\mu}\right]^{\frac{1}{2}}$$

$$\sigma_{22} = \frac{1}{6N}\left[3K + \left(L + \frac{32\pi H^2N^2T}{\mu}\right)^{\frac{1}{2}}\right]\exp\left[\frac{\mu K}{16\pi H^2N^2}\left\{\left(L + \frac{32\pi H^2N^2T}{\mu}\right)^{\frac{1}{2}} - L^{\frac{1}{2}}\right\}\right]$$

$$\sigma_{33} = \frac{1}{6N}\left[\left(L + \frac{32\pi H^2N^2T}{\mu}\right)^{\frac{1}{2}} - 3K\right]$$

$$\exp\left[-\frac{\mu K}{16\pi H^2N^2}\left\{\left(L + \frac{32\pi H^2N^2T}{\mu}\right)^{\frac{1}{2}} - L^{\frac{1}{2}}\right\}\right] \qquad \dots \quad (2.4)$$

the other components of the shear tensor σ_{ij} being zero. Hence

$$\sigma^2 = \frac{1}{18N^2 \exp{(2T)}} \left[2L + 9K^2 + \frac{64\pi H^2 N^2 T}{\mu} \right]. \quad \dots \quad (2.5)$$

The non vanishing components of the conformal curvature tensor are

$$C_{12}{}^{12} = C_{13}{}^{13} = -\tfrac{1}{2} C_{23}{}^{23} = \frac{1}{\exp{(2T)}} \left[\left(\frac{L^2 - K^2}{12N^2} \right) + \frac{4\pi H^2 (2T-1)}{3\mu} \right] \dots \quad (2.6)$$

The metric (1.25) has apparently a singularity at $T = -\dfrac{L}{a^2}$. However it is not a real singularity but it occurs as such on account of the coordinates chosen. The expressions for density, pressure, expansion, shear and the conformal curvature tensor remain finite for this value of T. The Kreschmann scalar $R_{hijk} R^{hijk}$ for this metric has the value

$$\frac{1}{4N^4} \exp(-4T)[a^2\{a^4 + 2K^2 - 2(a^2T + L)\}]$$

which is finite when $a^2T + L = 0$. It is also to be noted that as T tends to infinity this scalar as also the scalar of expansion, shear and the conformal curvature tensor tend to zero so that the metric tends to become flat. From the form of the metric (1.25) it is clear that the model exists during the time $T \geqslant T_1$ where $T_1 \geqslant -\dfrac{L}{a^2}$ and it has to be continued at time $T = T_1$ with another model valid during $T \leqslant T_1$ and representing the state prior to the evolution of the universe embodied by the metric (1.25).

In the absence of the magnetic field the model is given by the metric

$$ds^2 = N^2 \left[dX^2 - \frac{1}{L} \exp{(2T)} dT^2 \right] + \exp\left[\left(1 + \frac{K}{\sqrt{L}} \right) T \right] dY^2$$
$$+ \exp\left[\left(1 - \frac{K}{\sqrt{L}} \right) T \right] dZ^2 \quad \dots \quad (2.7)$$

for which the pressure p_0 and density ϵ_0 are given by

$$8\pi p_0 = \frac{1}{4N^2 \exp{(2T)}} (L - K^2) - \Lambda \quad \dots \quad (2.8)$$

$$8\pi \epsilon_0 = \frac{1}{4N^2 \exp{(2T)}} (L - K^2) + \Lambda \quad \dots \quad (2.9)$$

Also, for the metric (2.7)

$$C_{12}{}^{12} = C_{13}{}^{13} = -\tfrac{1}{2} C_{23}{}^{23} = \frac{(L^2 - K^2)}{12N^2 \exp{(2T)}}. \quad \dots \quad (2.10)$$

Thus, the magnetic field gives positive contribution to expansion, shear and the free gravitational field which die out for large values of T at a slower rate than the corresponding quantities in the absence of the magnetic field.

References

Marder L. 1958 *Proc. Roy. Soc.* **A246**, 133.

Heckmann O. and Schucking E. 1962 *Gravitation : An introduction to Current Research* ed. L. Witten, New York. 438.

Thorne K. S. 1967 *Astrophysics Jour.* **148**, 51.

Lichnerowicz A. 1967 *Relativistic Hydrodynamics and Magnetohydrodynamics.* W. A. Benzamin. Inc. New York. Amsterdam, 93.